where on the right-hand side we have introduced the SFS tensors associated to each flux, as defined above.

As in the nonrelativistic case, we propose to model the filtered SFS terms appearing in Eqs. (41)–(44) by means of a compact application of the gradient model. First of all, we define the *double gradient operator* H, acting on any given field X, as

$$H_X = H(X) \coloneqq \nabla \frac{dX}{d\bar{C}^b} \cdot \nabla \bar{C}^b, \tag{45}$$

which satisfies a sort of generalized Leibniz's rule,

$$H(XY) = XH(Y) + YH(X) + 2\nabla X \cdot \nabla Y. \tag{46}$$

Notice that, when acting on any conserved field, it vanishes by its definition (i.e., $H_D = H_S^i = H_U = H_B^i = 0$). On the other hand, when the operator applies to a nonconserved variable, the quantity is nonzero. This holds in particular for $\{\tilde{p}, \tilde{\Theta}, \tilde{v}^k\}$ as we shall see below.

By applying these rules and using Eq. (26), the SGS gradient tensors approximating the SFS terms of Eqs. (41)–(44) read

$$\tau_N^k = -\xi H_N^k, \qquad \tau_T^{ki} = -\xi H_T^{ki}$$
$$\tau_S^k = 0, \qquad \tau_M^{ki} = -\xi H_M^{ki}, \tag{47}$$

where the set of the H tensors, after some algebraic manipulations, can be written as[2]

$$H_N^k = 2\nabla\bar{D} \cdot \nabla\tilde{v}^k + \bar{D}H_v^k, \tag{48}$$

$$\begin{aligned} H_T^{ki} = {} & 2[\nabla\tilde{\mathcal{E}} \cdot \nabla(\tilde{v}^i\tilde{v}^k) + \tilde{\mathcal{E}}(\tilde{v}^{(i}H_v^{k)} + \nabla\tilde{v}^i \cdot \nabla\tilde{v}^k)] \\ & + \tilde{v}^i\tilde{v}^k H_{\mathcal{E}} - 2[\nabla\bar{B}^i \cdot \nabla\bar{B}^k + \nabla\tilde{E}^i \cdot \nabla\tilde{E}^k + \tilde{E}^{(i}H_E^{k)}] \\ & + \delta^{ki}[H_p + \nabla\bar{B}_j \cdot \nabla\bar{B}^j + \nabla\tilde{E}_j \cdot \nabla\tilde{E}^j + \tilde{E}_j H_E^j], \end{aligned} \tag{49}$$

$$H_M^{ki} = 4\nabla\bar{B}^{[i} \cdot \nabla\tilde{v}^{k]} + 2\bar{B}^{[i}H_v^{k]}, \tag{50}$$

where H_E^i is just the Hodge dual of H_M^{ij}, i.e., $H_E^i = \frac{1}{2}\epsilon^i{}_{jk}H_M^{jk}$. Notice the values of the double gradient appearing above, $\{H_p, H_\Theta, H_v^k, H_E^k, H_{\mathcal{E}}\}$, are meant to approximate the SFS residuals related to the nonconserved fields, $\{\bar{\tau}_p, \bar{\tau}_\Theta, \bar{\tau}_v^k, \bar{\tau}_E^k, \bar{\tau}_{\mathcal{E}}\}$, defined according to Eq. (21) as $\bar{\tau}_X \backsimeq -\xi H_X$ (exactly like for the conserved field SFS residuals). Their explicit expressions are obtained by computing the following set of equations in the order in which they appear, where the quantities $\tilde{\Psi}$ denote the auxiliary fields which are used to simplify the presentation (and also to facilitate their implementation):

$$\begin{aligned} \tilde{\Psi}_v^k &= \frac{2}{\tilde{\Theta}}\left\{\nabla(\tilde{v}\cdot\bar{B})\cdot\nabla\bar{B}^k - \nabla\tilde{\Theta}\cdot\nabla\tilde{v}^k + \frac{\bar{B}^k}{\tilde{\mathcal{E}}}[\tilde{\Theta}\nabla\bar{B}_j\cdot\nabla\tilde{v}^j + \bar{B}_j\nabla\bar{B}^j\cdot\nabla(\tilde{v}\cdot\bar{B}) - \bar{B}_j\nabla\tilde{v}^j\cdot\nabla\tilde{\Theta}]\right\}, \\ \tilde{\Psi}_M^{ki} &= \frac{4}{\tilde{\Theta}}[\tilde{\Theta}\nabla\bar{B}^{[i}\cdot\nabla\tilde{v}^{k]} + \bar{B}^{[i}\nabla\bar{B}^{k]}\cdot\nabla(\tilde{v}\cdot\bar{B}) - \bar{B}^{[i}\nabla\tilde{v}^{k]}\cdot\nabla\tilde{\Theta}], \\ \tilde{\Psi}_\Theta &= \frac{\tilde{\Theta}}{\tilde{\Theta}-\tilde{E}^2}\{\nabla\bar{B}_j\cdot\nabla\bar{B}^j - \nabla\tilde{E}_j\cdot\nabla\tilde{E}^j - \bar{B}_{[i}\tilde{v}_{k]}\tilde{\Psi}_M^{ki}\}, \\ \tilde{\Psi}_A &= \tilde{W}^2\left(\tilde{p}\frac{d\tilde{p}}{d\tilde{\epsilon}} + \tilde{\rho}^2\frac{d\tilde{p}}{d\tilde{\rho}}\right), \end{aligned}$$

$$\begin{aligned} \frac{H_{\mathrm{p}}}{\tilde{\Theta}-\tilde{E}^2} = {} & \frac{\tilde{\mathcal{E}}\tilde{W}^2}{(\tilde{\rho}\,\tilde{\mathcal{E}}-\tilde{\Psi}_A)(\tilde{\Theta}-\tilde{E}^2)\tilde{W}^2 + \tilde{\Psi}_A\tilde{\Theta}}\left\{\tilde{\rho}\left(\nabla\frac{d\tilde{p}}{d\tilde{\rho}}\cdot\nabla\tilde{\rho} + \nabla\frac{d\tilde{p}}{d\tilde{\epsilon}}\cdot\nabla\tilde{\epsilon}\right) - 2\frac{d\tilde{p}}{d\tilde{\epsilon}}\nabla\tilde{\rho}\cdot\nabla\tilde{\epsilon}\right. \\ & - \left(\tilde{\mathcal{E}}\frac{d\tilde{p}}{d\tilde{\epsilon}} - \tilde{\Psi}_A\right)\left[\frac{\tilde{W}^2}{4}\nabla\tilde{W}^{-2}\cdot\nabla\tilde{W}^{-2} + \nabla\tilde{W}^{-2}\cdot\nabla(\ln\tilde{\rho})\right] - \frac{2}{\tilde{W}^2}\frac{d\tilde{p}}{d\tilde{\epsilon}}[\nabla\bar{B}_j\cdot\nabla\bar{B}^j + \nabla\tilde{W}^2\cdot\nabla\tilde{h}] \\ & \left. - \left(\tilde{\mathcal{E}}\frac{d\tilde{p}}{d\tilde{\epsilon}} + \tilde{\Psi}_A\right)[\tilde{v}_k\tilde{\Psi}_v^k + \nabla\tilde{v}_j\cdot\nabla\tilde{v}^j + \tilde{W}^2\nabla\tilde{W}^{-2}\cdot\nabla\tilde{W}^{-2}] + \frac{1}{\tilde{\mathcal{E}}}\left[\left(\tilde{\mathcal{E}}\frac{d\tilde{p}}{d\tilde{\epsilon}} + \tilde{\Psi}_A\right)(\tilde{\Theta}-\tilde{E}^2) - \frac{\tilde{\Psi}_A\tilde{\Theta}}{\tilde{W}^2}\right]\frac{\tilde{\Psi}_\Theta}{\tilde{\Theta}}\right\}, \end{aligned} \tag{51}$$

[2]We use a mixed notation for scalar products when it comes about the gradients, i.e., $\nabla X \cdot \nabla Y$ instead of $\nabla_i X \nabla^i Y$, in order to make well visible the gradient terms, which are the core of the SGS model and always appear contracted to each other.